Spatial and statistical properties of the field scattered by random slightly rough surfaces

Propriétés spatiales et statistiques du champ diffusé par des surfaces aléatoires faiblement rugueuses

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Abstract

In the framework of the Small Perturbation Method, we define the spatial and statistical properties of the scattered field measured by a sensor. The scattering surface is defined as an infinite length random perturbation. In the intermediate-field zone, the scattered field is represented by a continuum of progressive plane waves. The sensor is a band-pass filter which keeps only progressive plane waves at angles within angular band $[\theta_1;\theta_2]$. If the angular band does not contain the specular angle, the measured field becomes an ergodic and stationary process.

Résumé

Dans le cadre de la méthode des petites perturbations, nous déterminons les propriétés statistiques et spatiales du champ diffusé mesuré par un capteur. La surface diffusante est représentée par une perturbation aléatoire d'extension infinie. En zone de champ intermédiaire, le champ diffusé est défini par un continuum d'ondes planes progressives. Le capteur est un filtre passe-bande qui ne retient que les ondes planes dont l'angle de diffusion appartient à la bande passante du capteur $[\theta_1;\theta_2]$. Si la bande passante ne contient pas l'angle de la direction spéculaire, le champ mesuré devient un processus stationnaire et ergodique à l'ordre 2.

Mots clès : Champ diffusé - Surface - Stationnarité - Ergodicité – Capteur

Keywords: Scattered field - Surface - Stationarity – Ergodicity – Sensor

1. Introduction

The problem of electromagnetic wave scattering from random surfaces continues to attract research interest because of its broad applications in optics, radio wave propagation and remote sensing. The three classical analytical methods commonly used in random rough-surface scattering are the small-perturbation method, the Kirchhoff method and the small slope approximation [1-3]. The electromagnetic analysis of rough surfaces with parameters close to the incident wavelength requires a rigorous formalism. Numerous methods based on Monte Carlo simulations are available for 1D and 2D random rough surfaces [4-8]. Most of research works focus on the determination of coherent and incoherent intensities. Nevertheless, there are various applications where the amplitude and phase statistical distributions of the field scattered from rough surfaces are required [9]. Consequently, it's important to define these statistical distributions from either an analytical approach or Monte-Carlo simulations.

In reference [10], we have proposed in the framework of the Small Perturbation Method an analytical study of the stationarity and the ergodicity of the field scattered by slightly rough random surfaces. Let us make a brief summary. The surfaces were assumed to be stationary and ergodic with a Gaussian height distribution and a zero mean value. First, in the intermediate-field zone we have demonstrated that the real part and the imaginary part of the scattered field are correlated Gaussian stochastic variables with zero mean values and unequal variances. For an infinite rough surface, we have shown that the scattered field amplitude is given by a Hoyt law and the phase is not uniformly distributed. We have also shown that under oblique incidence, the second order statistical moments and the statistical distributions characterizing the scattered field depend on the observation point. Consequently, the scattered field is not stationary. On the contrary, under normal incidence, the probability density functions do not depend on the observation point demonstrating that the scattered field represents a strictly stationary stochastic process. Second, for a given altitude, we have shown that the spatial variances are not stochastic variables and do not depend on the realization surface. Hence the scattered field is ergodic to the second order. Under oblique incidence, we have shown that the real part and the imaginary part of the scattered field are spatially uncorrelated and have equal variances. Therefore, the spatial distribution of the phase is uniform and the amplitude is given by a Rayleigh law. We have noticed that the spatial averages are different from the statistical ones. Under normal incidence, the real part and the imaginary part of the scattered field have different spatial variances and they are spatially correlated. Therefore, the phase of the scattered field associated with a single profile is not uniformly distributed and the amplitude distribution is given by a Hoyt law. Contrary to oblique incidence, the spatial and statistical distributions under normal incidence are interchangeable insofar as the scattered field represents an ergodic and stationary random process.

In this paper, we extend the study above to the measured scattered field and define the conditions that must verify the sensor transfer function so that the field is stationary and ergodic. In section 2, we present the spatial and statistical properties of the rough surfaces and give the scattered fields expressions in the framework of the Small Perturbation Method. In section 3, we recall the main results presented in reference [10]. Section 4 relates to the spatial and statistical characteristics of the measured scattered field. We establish the conditions that must verify the sensor transfer function so that the measured field is stationary and ergodic.

2. The scattered field as a random process

2.1 *The random surfaces under consideration*

We consider a random cylindrical surface illuminated by a monochromatic plane wave under incidence θ_0.

$$\psi_0(x,y) = \exp(-jk(\sin\theta_0 x + \cos\theta_0 y)) \quad \text{with} \quad k = 2\pi/\lambda \tag{1}$$

where λ is the wavelength. The incident wave vector lies in the xOy plane. Because of the 1D configuration of the problem, the incident field $\psi_0(x,y)$ and the scattered field $\psi(x,y)$ are represented by their Oz component only. For $E_{//}$ polarization, ψ_0 and ψ represent the Oz component of the electric vector and, for $H_{//}$ polarization, they represent the Oz component of the magnetic vector.

The surface separates the air from a dielectric material with a real or complex refractive index *n*. The surface is defined by equation $y = a(x)$. Hereafter, $A(x)$ represents the random process describing an infinite ensemble of surfaces and $a(x)$, a single realization of process. Let us assume the surface to be a stationary random Gaussian process with zero statistical mean value, with a statistical autocorrelation function $R_{AA}(x)$ that remains invariant of an arbitrary shift [10-11]. Assuming the stationarity of the Gaussian process, the statistical properties are independent of the absolute origin of the coordinate x. We also assume that the surface is an ergodic random process to second order with zero spatial mean value, and with the spatial autocorrelation $C_{aa}(x)$ that is a deterministic function independent on the realization. According to these assumptions and the Birkhoff theorem, we can identify the spatial autocorrelation function with the statistical autocorrelation function. Furthermore, we assume that the autocorrelation function has a finite memory (i.e., $R_{AA}(x)$ vanishes as $x \to \pm\infty$). Hereafter, $\hat{R}_{AA}(\alpha)$ is the Fourier Transform of $R_{AA}(x)$ and represents the power spectral density of the random process $A(x)$.

We emphasize that we distinguish the concepts of ergodicity and stationarity. It is said that a process is ergordic to the second order if the spatial averaging until the second order are independent of the choice of the realization. A random process is stationary to the second order (wide-sense stationary) if the statistical averaging until the second order remains invariant of an arbitrary space shift. As a consequence, a random process can be ergodic to the second order but not stationary and inversely [11]. Nevertheless, for a stationary and ergodic process, the statistical averaging and the spatial averaging are interchangeable.

2.2 *The scattered field*

Let us define the scattered field $\psi(x)$ as the total field minus the sum of the incident field and the field reflected by the mean plane of the surface. Above the surface, the scattered field can be represented by a superposition of a continuous spectrum of outgoing plane waves [3], the so-called Rayleigh integral. More precisely, beyond a determined height above the surface where evanescent waves can be neglected, the Rayleigh integral is reduced to the contribution of the propagating waves $(|\alpha| \le k)$ only. In the framework of the first order Small Perturbation Method [4], the scattered field can be expressed by

$$\psi(x,y) = \frac{1}{2\pi}\int_{-k}^{k} \hat{K}(\alpha)\, \hat{a}(\alpha - \alpha_0) e^{-j(\alpha x + \beta y)} d\alpha \tag{2}$$

with

$$\hat{a}(\alpha) = \int_{-L/2}^{L/2} a(x) e^{j\alpha x} dx \tag{3}$$

and

$$\alpha = k \sin\theta \quad ; \quad \beta = \sqrt{k^2 - \alpha^2} \quad ; \quad \alpha_0 = k \sin\theta_0 \tag{4}$$

The time-dependence factor $\exp(j\omega t)$, where ω is the angular frequency, is omitted. The function $\hat{a}(\alpha)$ may be considered as the Fourier transform of the limited profile $a(x)$ over the interval of finite length L. For the study of the spatial and statistical properties of the scattered field, the profile will be then extended to infinity $(L \to \infty)$. Hereafter, $\psi(x,y)$ represents the field scattered by the limited profile $a(x)$ and $\Psi(x,y)$, the spatial stochastic process describing an infinite ensemble of solutions $\psi(x,y)$.

Obviously, $\hat{A}(\alpha)$ contains the random feature of the random process $\Psi(x,y)$. $\hat{K}(\alpha)$ is a deterministic function depending on the polarization, the observation angle, the incidence angle and the refractive index. Since (3) is a linear transformation of a Gaussian process $A(x)$, the real part $\Psi_r = \mathrm{Re}(\Psi)$ and the imaginary part $\Psi_i = \mathrm{Im}(\Psi)$ are also quantities distributed with Gaussian probability density functions.

3. Spatial and statistical properties of the first-order scattered field

3.1 *Statistical properties*

In reference [10], we have shown that, the statistical mean of the scattered field in first-order is null at any point $M(x,y)$ and that if $L \to +\infty$, the real part Ψ_r and the imaginary part Ψ_i are Gaussian random processes whose the statistical variances $\langle \Psi_r^2 \rangle$ and $\langle \Psi_i^2 \rangle$ and the statistical covariance $\langle \Psi_r \Psi_i \rangle$ are

$$\begin{aligned} <\Psi_r^2> &= V_1 + \mathrm{Re}(V_2) \\ <\Psi_i^2> &= V_1 - \mathrm{Re}(V_2) \\ <\Psi_r \Psi_i> &= \frac{1}{2} \mathrm{Im}(V_2) \end{aligned} \tag{5}$$

where

$$V_1 = \frac{1}{4\pi} \int_{-k}^{k} \left| \hat{K}(\alpha) \right|^2 \hat{R}_{AA}(\alpha - \alpha_0) d\alpha \tag{6}$$

and

$$V_2 = e^{-2j\alpha_0 x} \int_{-k}^{k} \hat{K}(\alpha) \hat{K}(2\alpha_0 - \alpha) \hat{R}_{AA}(\alpha - \alpha_0) e^{-j(\beta(\alpha) + \beta(\alpha - 2\alpha_0))y} d\alpha \tag{7}$$

Expressions (5) to (7) show that the real part and the imaginary part are correlated, $\langle \Psi_r \Psi_i \rangle \neq 0$, with unequal variances $\langle \Psi_r^2 \rangle \neq \langle \Psi_i^2 \rangle$. These statistical parameters depend on the observation point except at normal incidence ($\alpha_0 = 0$) where they depend on the altitude *y* only. As a consequence, the scattered field is not stationary to the second order.

Since real and imaginary parts of the scattered field are Gaussian centered processes, the probability density functions of the amplitude $|\Psi|$ and the probability law of the phase Φ are defined as follows

$$p_{|\Psi|}(m) = \frac{m}{\sigma_r \sigma_i \sqrt{1-\rho^2}} \mathrm{I}_0\left(\frac{m^2}{4(1-\rho^2)} \sqrt{\frac{(\sigma_i^2 - \sigma_r^2)^2}{\sigma_r^4 \sigma_i^4} + \frac{4\rho^2}{\sigma_r^2 \sigma_i^2}} \right) e^{-\frac{m^2}{4\sigma_r^2\sigma_i^2}\frac{\sigma_r^2+\sigma_i^2}{1-\rho^2}} \tag{8}$$

$$p_\Phi(\phi) = \frac{1}{2\pi} \frac{\sigma_r \sigma_i \sqrt{1-\rho^2}}{\sigma_i^2 \cos^2\phi - \rho\sigma_r\sigma_i \sin 2\phi + \sigma_r^2 \sin^2\phi} \tag{9}$$

where

$$\sigma_r = \sqrt{\langle \Psi_r^2 \rangle} \;;\; \sigma_i = \sqrt{\langle \Psi_i^2 \rangle} \;;\; \rho = \langle \Psi_r \Psi_i \rangle / \sigma_r \sigma_i \tag{10}$$

ρ is the correlation coefficient and I_0 is the zeroth-order modified Bessel function [13]. The scattered field amplitude is given by a Hoyt law (8) and the phase is not uniformly distributed (9). From these expressions, we obtain the uniform law and the Rayleigh law if $\sigma_r = \sigma_i$ and $\rho = 0$. We note that under oblique incidence, the variances σ_r^2 and σ_i^2, the correlation coefficient ρ and the probability laws depend on the coordinate x. On the contrary, under normal incidence, the probability density functions do not depend on the coordinate x and as a consequence, the scattered field represents a strictly stationary stochastic process.

3.2 *Spatial properties*

We consider the field $\psi(x,y)$ scattered by a single realization $a(x)$. In reference [10], we have shown that the spatial mean of the scattered field along a horizontal line is zero and that the spatial variances $\overline{\psi_r^2}$, $\overline{\psi_i^2}$ and the spatial covariance $\overline{\psi_r \psi_i}$ associated with real and imaginary parts of the scattered field are given as follows

$$\begin{aligned} \overline{\psi_r^2} &= V_1 + \mathrm{Re}(V_2)\delta_{\alpha,\alpha_0} \\ \overline{\psi_i^2} &= V_1 - \mathrm{Re}(V_2)\delta_{\alpha,\alpha_0} \\ \overline{\psi_r \psi_i} &= \frac{1}{2}\mathrm{Im}(V_2)\delta_{\alpha,\alpha_0} \end{aligned} \tag{11}$$

where δ_{α,α_0} represents the Kronecker symbol. Terms V_1 and V_2 are given by Equations (6) and (7). Since the spatial variances do not depend on the realization, the scattered field is an ergodic centered process to the second order. Under oblique incidence, the spatial variances are equal and independent on the line altitude. The real part ψ_r and the imaginary part ψ_i are spatially uncorrelated. Under normal incidence, the spatial variances are different. The real part ψ_r and the imaginary part ψ_i are spatially correlated. Under normal incidence, the scattered field is an ergodic and stationary random process. For this reason, contrary to oblique incidence, the spatial and statistical variances under normal incidence are interchangeable.

We can also define the spatial distributions associated with the scattering field using (8) and (9) with spatial averages instead of statistical averages. Under oblique incidence, the real part and the imaginary part of the scattering field are spatially uncorrelated and have the same spatial variance. Therefore, the phase is spatially uniformly distributed and the amplitude is given by a Rayleigh law. Under normal incidence, the real part and the imaginary part have different spatial variances and are spatially correlated. The phase of the scattered field $\psi(x,y)$ associated with the single profile $a(x)$ is not uniformly distributed and the amplitude distribution is given by a Hoyt law. Contrary to oblique incidence, the spatial and statistical distributions under normal incidence are interchangeable.

4. Spatial and statistical properties of the measured scattered field

The measured scattered field $\psi_m(x,y)$ is given by the following expression

$$\psi_m(x,y) = \frac{1}{2\pi}\int_{\alpha_1}^{\alpha_2} \hat{K}(\alpha)\,\hat{F}(\alpha)\hat{a}(\alpha-\alpha_0)e^{-j\alpha x+\beta(\alpha)y}d\alpha \qquad (12)$$

where $\hat{F}(\alpha)$ represents the sensor transfer function. The sensor is a band-pass filter which keeps only progressive plane waves of the scattered field within the angular band $[\theta_1;\theta_2]$:

$$\begin{aligned}&\hat{F}(\alpha)\neq 0 \text{ if } \alpha_1<\alpha<\alpha_2\\ &\hat{F}(\alpha)=0 \text{ elsewhere}\end{aligned} \qquad (13)$$

with

$$\alpha_1 = k\sin\theta_1 \text{ and } \alpha_2 = k\sin\theta_2 \qquad (14)$$

We can derive the statistical and spatial properties of the measured scattered field using relations (5) to (11) with $\hat{K}(\alpha)\hat{F}(\alpha)$ instead of $\hat{K}(\alpha)$. The measured field is an ergodic centered process and represents a stationary process under normal incidence only. The measured field becomes a stationary process if the second order moments are independent on the x coordinate. From Eqs (6) to (8), we can deduce that the statistical variances do not depend on x if $V_2=0$. This case occurs if

$$\hat{F}(2\alpha_0-\alpha)=0 \text{ for } \alpha_1<\alpha<\alpha_2 \qquad (15)$$

We deduce that the measured field is stationary if

$$\alpha_0<\alpha_1<\alpha_2 \quad \text{or} \quad \alpha_1<\alpha_2<\alpha_0 \qquad (16)$$

If condition (16) is proved, the measured field represents an ergodic and stationary centered Gaussian process. In this case, all the variances are equal and independant on the coordinate y

$$<\Psi_{r,i}^2> = \overline{\psi_{r,i}^2} = \frac{1}{4\pi}\int_{-k}^{k}\left|\hat{K}(\alpha)\right|^2 \hat{R}_{AA}(\alpha-\alpha_0)d\alpha \qquad (17)$$

Moreover, the real part and the imaginary part of the measured field are statistically and spatially uncorrelated

$$\overline{\psi_r\psi_i} = <\Psi_r\Psi_i> = 0 \qquad (18)$$

Under condition (16), the statistical and spatial distributions are interchangeable. Since the real part and the imaginary part have the same variances and since they are uncorrelated, the phase is uniformly distributed and the amplitude obeys to a Rayleigh law.

5. Conclusion

In reference [10], we have shown that the field scattered by slightly rough random surfaces illuminated under oblique incidence θ_0 is ergodic but not stationary. In this paper, we have defined the condition that must verify the sensor transfer function so that the measured scattered field in the intermediate-field zone becomes an ergodic and stationary process. The sensor is a band-pass filter which keeps only progressive plane waves of the scattered field within angular band $[\theta_1;\theta_2]$. The measured scattered field becomes an ergodic and stationary process if the angular band does not contain the specular angle. With more complex manipulations, this analytical can be extended to the two-dimensional case.

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